THE TOTAL CHROMATIC NUMBER OF GRAPHS OF HIGH MINIMUM DEGREE

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ABSTRACT

If G is a simple graph with minimum degree $\delta(G)$ satisfying $\delta(G) \geqslant \frac{5}{6}(|V(G)|+1)$ the total chromatic number conjecture holds; moreover if $\delta(G) \geqslant \frac{3}{4}|V(G)|$ then $\chi_T(G) \leqslant \Delta(G)+3$. Also if G has odd order and is regular with $d(G) \geqslant \frac{1}{3}\sqrt{7}|V(G)|$ then a necessary and sufficient condition for $\chi_T(G) = \Delta(G)+1$ is given.

1. *Introduction*

The graphs in this paper will be simple—that is, they will have no loops or multiple edges. An *edge-colouring* of a graph G is a map $\phi: E(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours, such that no two adjacent edges receive the same colour. The least value of $|\mathscr{C}|$ for which G has an edge-colouring with $|\mathscr{C}|$ colours is the *chromatic index* (or edge-chromatic number) $\chi'(G)$ of G. A famous result of Vizing [19] states that, for a graph G, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. If $\chi'(G) = \Delta(G)$, then G is of *class* 1, and if $\chi'(G) = \Delta(G)+1$, then G is of *class* 2. In [6] the first two authors showed that, if G is a regular graph of even order satisfying $d(G) \geqslant \frac{6}{7}|V(G)|$, where $d(G)$ denotes the common degree of the vertices of a regular graph, then G is of class 1. In [7] they gave another proof of this result, placing it in a somewhat wider context. In [8] they improved the bound to $d(G) \geqslant \frac{1}{2}(\sqrt{7}-1)|V(G)|$, and in [14] the second author generalized it even further.

A *vertex-colouring* of a graph G is a map $\psi: V(G) \to \mathscr{C}$, such that no two adjacent vertices receive the same colour.

A *total-colouring* of a graph G is a map $\theta: E(G) \cup V(G) \to \mathscr{C}$, such that no two incident or adjacent elements of G receive the same colour. The least value of $|\mathscr{C}|$ for which G has a total-colouring with $|\mathscr{C}|$ colours is the *total chromatic number* $\chi_T(G)$ of G. A long-standing and notoriously difficult conjecture of Behzad [2] is that $\Delta(G)+1 \leqslant \chi_T(G) \leqslant \Delta(G)+2$ (of course the lower bound is obvious). Several quite good upper bounds for the total chromatic number have been discovered recently. The first of these, due to Hind [15], is that

$$\chi_T(G) \leqslant \chi'(G)+2\lceil\sqrt{\chi(G)}\rceil,$$

where $\chi(G)$ and $\chi'(G)$ are the chromatic number and the chromatic index of G respectively. The second, due to Chetwynd and Häggkvist [5], is that if $t! > |V(G)|$ then

$$\chi_T(G) \leqslant \chi'(G)+t.$$

The third, also due to Hind [16], is that

$$\chi_T(G) \leqslant \Delta(G)+2\left\lceil\frac{|V(G)|}{\Delta(G)}\right\rceil+1.$$

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We perhaps should mention here that earlier bounds on the list chromatic number were obtained which gave as a bonus bounds on the total chromatic number. The earliest of these was due to Bollobás and Harris [4], and their result yielded the bound $\chi_T(G) \leqslant \frac{11}{6}\Delta(G) + o(\Delta(G))$.

If $\chi_T(G) = \Delta(G)+1$, then we shall say that G is of *type* 1, and if $\chi_T(G) = \Delta(G)+2$, then we shall say that G is of *type* 2. No graphs have been shown to be of other than type 1 or type 2, which supports Behzad's conjecture. Let $\delta(G)$ denote the minimum degree of a graph. In this paper we characterize the regular graphs of odd order satisfying $d(G) \geqslant \frac{1}{3}\sqrt{7}|V(G)|$ which are of type 1 and those which are of type 2. We also show, amongst other things, that the total chromatic number conjecture is true for graphs G satisfying $\delta(G) \geqslant \frac{5}{6}(|V(G)|+1)$ and that if $\delta(G) \geqslant \frac{3}{4}|V(G)|$ then $\chi_T(G) \leqslant \Delta(G)+3$. In [14] the second author recently evaluated $\chi_T(G)$ when G is a graph of even order with a spanning star. It is beginning to seem that the total chromatic number is amenable to study, even in the absence of an analogue of Vizing's theorem.

We remark that the lower bounds on $\delta(G)$ in all our theorems reflect nothing more than the limitations of our method of proof. We know of nothing to suggest that the 'correct' lower bounds should not in every case be much lower still.

Finally we remark that there is a lot of difference in the results we have been able to prove between the cases when $|V(G)|$ is even and when it is odd. To reflect this, the two cases are treated in separate sections.

2. *Preliminary results*

In [14] the second author proved the following.

LEMMA 1. *Let G be a graph of even order, $|V(G)| = 2n$. If G has r vertices of maximum degree, and $\delta(G)$, the minimum degree, satisfies*

$$\delta(G) \geqslant n+r+4$$

then G is of class 1.

In [8] the first two authors proved the following.

LEMMA 2. *Let G be a regular graph of degree $d(G)$ and of even order satisfying*

$$d(G) \geqslant \tfrac{1}{2}(\sqrt{7}-1)|V(G)|.$$

Then G is of class 1.

The next lemma is an extension due to Berge [3] of a well-known theorem of Chvátal [10].

LEMMA 3. *Let G be a graph of order n and degrees $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$. Let q be an integer, $0 \leqslant q \leqslant n-3$. If, for every k with $q < k < \frac{1}{2}(n+q)$, the following condition holds:*

$$d_{k-q} \leqslant k \Rightarrow d_{n-k} \geqslant n-k+q$$

then, for each set F of independent edges with $|F| = q$, there exists a Hamiltonian cycle containing F.

The next lemma is due to Erdös and Posá [**11**].

LEMMA 4. *A simple graph G contains a matching of size at least*

$$\min\{\delta(G), \lfloor\tfrac{1}{2}|V(G)|\rfloor\}.$$

3. *Graphs of odd order*

In this section we characterize type 1 and type 2 regular graphs of odd order with $d(G) \geqslant \frac{1}{3}\sqrt{7}|V(G)|$. We also show that if $\delta(G) \geqslant \frac{5}{6}(|V(G)|+1)$ then $\chi_T(G) \leqslant \Delta(G)+2$ and that if $\delta(G) \geqslant \frac{3}{4}|V(G)|$ then $\chi_T(G) \leqslant \Delta(G)+3$. We also characterize type 1 and type 2 regular graphs of odd order with $d(G) = |V(G)|-3$.

THEOREM 1. *Let G be a graph of odd order with*

$$\delta(G)+2\Delta(G) \geqslant \tfrac{5}{2}(|V(G)|+1).$$

Then

$$\chi_T(G) \leqslant \Delta(G)+2.$$

COROLLARY. *Let G be a graph of odd order with*

$$\delta(G) \geqslant \tfrac{5}{6}(|V(G)|+1).$$

Then

$$\chi_T(G) \leqslant \Delta(G)+2.$$

Proof of Theorem 1. Let $|V(G)| = 2n+1$ for some $n \geqslant 1$. In this proof, let $\delta = \delta(G)$ and $\Delta = \Delta(G)$, and let the vertices of G be $v_1, \ldots, v_{2n+1}$. For $1 \leqslant j \leqslant 2n-\Delta$, let the pair v_j, $v_{(2n-\Delta)+j}$ be non-adjacent. It follows by applying Lemma 4 to the complement of G that $2n-\Delta$ such pairs of vertices exist.

From G form a graph G^* by introducing a vertex v^* and joining it to $v_{2n-\Delta+1}, \ldots, v_{2n+1}$. Then G^* has $\Delta+2$ vertices, $v_{2n-\Delta+1}, \ldots, v_{2n+1}, v^*$, of degree at most $\Delta(G)+1$, and $2n-\Delta$ vertices, $v_1, \ldots, v_{2n-\Delta}$, of degree at most $\Delta(G)$. Since $\Delta \geqslant \frac{5}{6}(|V(G)|+1) \geqslant \frac{1}{3}(4n-1)$, it follows that $2n+1-2(2n-\Delta) \geqslant 2n-\Delta$. Therefore, for $1 \leqslant j \leqslant 2n-\Delta$, there is a vertex $v_{j+2(2n-\Delta)}$.

Let $F_1, \ldots, F_{2n-\Delta}$ be edge-disjoint matchings of G^* such that, for $1 \leqslant j \leqslant 2n-\Delta$, F_j misses the two vertices v_j and $v_{j+2(2n-\Delta)}$, contains the edge $v_{j+(2n-\Delta)}v^*$, and misses no further vertex. We show now that these matchings do exist.

We pick out these matchings one by one. For $1 \leqslant j \leqslant 2n-\Delta$, suppose that $F_1, \ldots, F_{j-1}$ have been constructed. Let $G_j^* = G^*\backslash(F_1 \cup \ldots \cup F_{j-1})$ and let G_j^+ be the simple graph formed from G_j^* by adding in the edges $v_j v_{2(2n-\Delta)+j}$ and $v_{(2n-\Delta)+j}v_{2(2n-\Delta)+j}$ if they are not already in G_j^*. Each vertex of G_j^+ has degree at least

$$\delta-(j-1) \geqslant \delta-(2n-\Delta)+1 = \delta+\Delta-2n+1.$$

Since $\delta+2\Delta \geqslant \frac{5}{2}(2n+1)+\frac{5}{2}$, it follows that

$$\delta+\Delta-2n+1 \geqslant \tfrac{3}{2}(2n+1)+\tfrac{5}{2}-2n+1 > n+3 > \tfrac{1}{2}((2n+2)+3).$$

Therefore by Lemma 3, the graph G_j^+ contains a Hamiltonian cycle containing the path $v_j, v_{2(2n-\Delta)+j}, v_{(2n-\Delta)+j}, v^*$. Therefore G_j^* contains a matching F_j which contains the edge $v^*v_{j+(2n-\Delta)}$, misses the vertices v_j and $v_{j+2(2n-\Delta)}$, and misses no other vertex. Iterating this gives the required matchings $F_1, \ldots, F_{2n-\Delta}$.

Let $G^{**} = G^{*}\backslash(F_1 \cup \ldots \cup F_{2n-\Delta})$. Then G^{**} has $2n-\Delta$ vertices $v_{2(2n-\Delta)+1}, \ldots, v_{3(2n-\Delta)}$ of degree at most $\Delta+1-(2n-\Delta-1) = 2\Delta-2n+2$, and the remaining vertices have degree at most $2\Delta-2n+1$. Since $\delta+2\Delta \geqslant \frac{5}{2}(2n+1)+\frac{5}{2}$, it follows that

$$\delta(G^{**}) \geqslant \delta+\Delta-2n \geqslant (n+1)+(2n-\Delta)+4,$$

and so it follows from Lemma 1 and Vizing's theorem that G^{**} can be edge-coloured with $2\Delta-2n+2$ colours.

From an edge-colouring of G^{**} with the $2\Delta-2n+2$ colours $c_{2n-\Delta+1}, \ldots, c_{\Delta+2}$, we form a total-colouring of G with the $\Delta+2$ colours $c_1, \ldots, c_{\Delta+2}$ as follows. Each edge of G which is also an edge of G^{**} receives the same colour. For

$$2(2n-\Delta)+1 \leqslant j \leqslant 2n+1,$$

the vertex v_j receives the colour of the edge $v_j v^*$. For $1 \leqslant j \leqslant 2n-\Delta$, the two vertices v_j and $v_{2n-\Delta+j}$ and the edges of $F_j\backslash\{v^* v_{(2n-\Delta)+j}\}$ all receive the colour c_j. It is easy to check that this is a total-colouring of G.

This proves Theorem 1.

Let $\operatorname{def}(G) = \sum_{v\in V(G)} (\Delta(G)-d_G(v))$. If G has a vertex-colouring with $\Delta(G)+1$ colours such that the number of colour classes with parity different from $|V(G)|$ is at most $\operatorname{def}(G)$, then we call G *conformable*. It was shown in [9] that if G is type 1, then G is conformable, but the converse need not be true. Chetwynd and Hilton [9] made the following conjecture.

CONJECTURE 1. Let $\Delta(G) \geqslant \frac{1}{2}(|V(G)|+1)$. Then G is of type 1 if and only if every subgraph H of G with $\Delta(H) = \Delta(G)$ is conformable.

It is fairly easy to see that if G is of type 1, then G has no such non-conformable subgraph. The graph $K_{n,n}$ with n even is conformable but of type 2, so the figure $\Delta(G) \geqslant \frac{1}{2}(|V(G)|+1)$ cannot be made lower (if n is odd, then $K_{n,n}$ is not conformable).

In our next theorem we provide evidence for this conjecture by characterizing regular type 1 graphs of odd order with degree, $d(G)$, satisfying $d(G) \geqslant \frac{1}{3}\sqrt{7}|V(G)|$.

THEOREM 2. *Let G be a regular graph of odd order $|V(G)| = 2n+1$ with*

$$d(G) \geqslant \tfrac{1}{3}\sqrt{7}|V(G)|.$$

Then G is of type 1 *if and only if every subgraph H of G with $\Delta(H) = \Delta(G)$ is conformable. Otherwise G is of type* 2. (Note that $\frac{1}{3}\sqrt{7} \simeq 0.883$.)

In [6] we made the conjecture that if G is a regular graph of even order, and if $d(G) \geqslant \frac{1}{2}|V(G)|$, then G is of class 1. In view of Conjecture 1, it seems most probable that Theorem 2 also is true if $d(G) \geqslant \frac{1}{2}|V(G)|$. If that is the case, then there is an unexpected, interesting aspect to it. For graphs of even order, it follows from Turán's theorem that if $d(G) \geqslant \frac{1}{2}|V(G)|$, then $\bar{G}$ need not contain a K_3. This is true if G has even order, even if G is regular. However if G is regular and has odd order then G is non-bipartite and thus contains an odd circuit. It follows (as we show in Lemma 5 below) that if $\frac{1}{2}|V(G)| \leqslant d(G) \leqslant \frac{3}{5}|V(G)|$, then $\bar{G}$ will contain a K_3, and in fact it seems likely that it must contain many (vertex) disjoint graphs K_3 (although so far as we are aware, this has not been established); now if $\bar{G}$ contains $\frac{1}{2}(|V(G)|-d-1)$ (vertex) disjoint K_3 then G has a vertex-colouring with

$$\tfrac{1}{2}(|V(G)|-d-1)+\{|V(G)|-\tfrac{3}{2}(|V(G)|-d-1)\} = d+1$$

colours, each colour occurring on an odd number of vertices, so G is conformable. Thus it would follow from the condition in Theorem 2 that G is of type 1. Thus it seems very probable that if $\frac{1}{2}|V(G)| \leqslant d(G) \leqslant \frac{3}{5}|V(G)|$, and if G is regular of odd order, then G must be of type 1. The fact that $\bar{G}$ will contain a K_3 follows from the following easy lemma (which was kindly communicated to the authors by Dr J. Sheehan; see also [**18**]).

LEMMA 5. *If G is a regular graph whose shortest odd-cycle length is l, then*

$$d(G) \leqslant \frac{2}{l}|V(G)|.$$

Proof. Let C be a cycle of length l. Then each vertex not on C is joined to at most two vertices of C, and thus there are at most $2(|V(G)|-l)$ further edges incident with the vertices of C. But each vertex of C has $d(G)-2$ further edges incident with it. Thus

$$l(d(G)-2) \leqslant 2(|V(G)|-l),$$

from which the lemma follows.

Theorem 2 follows easily from the following lemma.

LEMMA 6. *Let G be a regular graph of odd order $|V(G)| = 2n+1$ satisfying*

$$d = d(G) \geqslant \tfrac{1}{3}\sqrt{7}|V(G)|.$$

Let $\bar{G}$ contain a subgraph $K_{i_1} \cup \ldots \cup K_{i_s}$, where $K_{i_1}, \ldots, K_{i_s}$ are vertex-disjoint complete graphs of odd orders $i_1, \ldots, i_s$, respectively, with $i_j \geqslant 3$ $(1 \leqslant j \leqslant s)$ and

$$(i_1+\ldots+i_s)-s = 2n-d.$$

Then G is of type 1.

REMARK. Since it could well be that $i_j = i_{j'}$ for some $j \neq j'$ the reader may well feel that the notation here is unsatisfactory; however it is convenient, and we crave his or her indulgence.

Proof. Since $|V(G)|$ is odd, it follows that $d = d(G)$ is even. Let $i_1 \geqslant \ldots \geqslant i_s$. For $1 \leqslant j \leqslant s$, let $v_{j,1}, \ldots, v_{j,i_j}$ be the vertices of K_{i_j}. Let $x = \frac{1}{2}(2n-d)$ and let

$$\{v_1, \ldots, v_{2x}\} = \bigcup_{j=1}^{s} \{v_{j,1}, \ldots, v_{j,i_j-1}\}.$$

Let $v_{1,i_1} = v_{2x+1}, \ldots, v_{s,i_s} = v_{2x+s}$ and let the remaining vertices of $V(G)$ be $v_{2x+s+1}, \ldots, v_{2n+1}$. From G form a graph G^* by introducing a vertex v^* and joining it to each vertex of $v_{2x+1}, \ldots, v_{2n+1}$. Then v^* and $v_{2x+1}, \ldots, v_{2n+1}$ each have degree $d+1$ in G, and $v_1, \ldots, v_{2x}$ each have degree d.

Now let $F_1, \ldots, F_x$ be edge-disjoint matchings of G such that, for $1 \leqslant j \leqslant s$, the vertices missed by F_j are precisely $v_{j,1}, \ldots, v_{j,i_j-1}$, and F_j includes the edge v^*v_{2x+j}; also let $F_{s+1}, \ldots, F_x$ be 1-factors.

We show now that G^* includes such a set of matchings. For $1 \leqslant j \leqslant x$, suppose that $F_1, \ldots, F_{j-1}$ have been constructed. If $1 \leqslant j \leqslant s$, we consider the graph G_j^+ obtained from $G^* \backslash \{F_1 \cup \ldots \cup F_{j-1}\}$ by joining $v_{j,k}$ to $v_{j,k+1}$ $(k = 1, 2, \ldots, i_j - 1)$. If $s+1 \leqslant j \leqslant x$ we let $G_j^+ = G^* \backslash \{F_1 \cup \ldots \cup F_{j-1}\}$. Then

$$\delta(G_j^+) \geqslant d-(j-1) = d-j+1 \geqslant d-\tfrac{1}{2}(2n-d)+1 = \tfrac{3}{2}d-n+1.$$

Since $d \geqslant \frac{1}{3}\sqrt{7}|V(G)| \geqslant \frac{3}{2}n+\frac{1}{4}$, it follows that

$$\delta(G_j^+) \geqslant \tfrac{3}{2}d-n+1 \geqslant 2n-\tfrac{1}{2}d+\tfrac{3}{2} = \tfrac{1}{2}((2n+2)+2x+1) \geqslant \tfrac{1}{2}((2n+2)+i_j).$$

Consequently, by Lemma 3, G_j^+ has a Hamiltonian cycle containing the path $v_{j,1}, v_{j,2}, \ldots, v_{j,i_j}, v^*$. Therefore $G^* \backslash \{F_1 \cup \ldots \cup F_{j-1}\}$ contains the required matching F_j. Iterating this shows that the matchings $F_1, \ldots, F_x$ do exist.

Let $G^{**} = G^* \backslash \{F_1 \cup \ldots \cup F_x\}$. Then G^{**} is regular of degree

$$d+1-x = \tfrac{3}{2}d-n+1.$$

Since $d \geqslant \frac{1}{3}\sqrt{7}|V(G)|$ it follows that

$$d(G^{**}) = \tfrac{3}{2}d-n+1 \geqslant \tfrac{3}{2}\cdot\tfrac{1}{3}\sqrt{7}|V(G)|-n+1 \geqslant \tfrac{1}{2}(\sqrt{7}-1)(2n+2).$$

Therefore by Lemma 2, G^{**} is of class 1.

Note that $x+(\frac{3}{2}d-n+1) = d+1$. From an edge-colouring of G^{**} with the $\frac{3}{2}d-n+1$ colours $c_{x+1}, \ldots, c_{d+1}$ we can obtain a total-colouring of G with the $d+1$ colours $c_1, \ldots, c_{d+1}$ as follows. Each edge of G which is also an edge of G^{**} receives the same colour. For $2x+s+1 \leqslant j \leqslant 2n+1$, the vertex v_j receives the colour of the edge $v_j v^*$. For $1 \leqslant j \leqslant s$, the i_j vertices $v_{j,1}, \ldots, v_{j,i_j}$ each receive the colour c_j. For $1 \leqslant j \leqslant x$, the edges of $F_j \backslash \{v_j v^*\}$ receive the colour c_j. It is easy to check that this is a total-colouring of G.

This proves Lemma 6.

Proof of Theorem 2. Suppose that G is a regular graph of odd order satisfying $d(G) \geqslant \frac{1}{3}\sqrt{7}|V(G)|$. Then, since $\frac{5}{6} \leqslant \frac{1}{3}\sqrt{7}$, it follows from Theorem 1 that $\chi_T(G) \leqslant d(G)+2$, and so G is of type 1 or type 2.

Now suppose that G is conformable. Then G has a vertex-colouring with $d(G)+1$ colours $c_1, \ldots, c_{d(G)+1}$ in which each colour class is odd. For $1 \leqslant j \leqslant d(G)+1$, let the number of vertices coloured c_j be i_j. We may assume that, for some s,

$$i_1 \geqslant i_2 \geqslant \ldots \geqslant i_s > i_{s+1} = \ldots = i_{d+1} = 1.$$

Then $(i_1+\ldots+i_s)-s = 2n-d$, and $\bar{G}$ contains a subgraph $K_{i_1} \cup \ldots \cup K_{i_s}$, where $K_{i_1}, \ldots, K_{i_s}$ are vertex disjoint complete graphs of odd orders $i_1, \ldots, i_s$. By Lemma 6, therefore, G is of type 1.

Conversely, suppose that G is of type 1. Then, as is shown by an easy argument in **[9]**, G is conformable.

This proves theorem 2.

We now mention a result of Fournier **[12]** (which is also an immediate consequence of Vizing's adjacency lemma **[21]**). Let G_Δ denote the subgraph of G induced by the vertices of maximum degree.

LEMMA 7. *If G_Δ is a forest, then G is of class* 1.

THEOREM 3. *Let G be a regular graph of odd order, $|V(G)| = 2n+1 \geqslant 3$, with $d(G) = 2n-2$. Then G is of type* 1 *if $\bar{G}$ contains a K_3, and is of type* 2 *otherwise.*

Proof. The theorem is easy to check if $n = 1$ or 2. From now on let $n \geqslant 3$.

Suppose first that $\bar{G}$ contains a K_3. Let the vertices of this K_3 be v_1, v_2 and v_3. For $n \geqslant 4$ we have

$$d = 2n-2 \geqslant \tfrac{1}{2}((2n+1)+2),$$

so, by Lemma 3, it follows that $G \cup \{v_1 v_2, v_2 v_3\}$ has a Hamiltonian cycle, and so G contains a matching F which misses v_1, v_2 and v_3, but misses no other vertex. If $n = 3$ it is easy to see directly that F exists. The graph $G \backslash F$ has three non-adjacent vertices of maximum degree, and so, by Lemma 7, it is of class 1. Let $G \backslash F$ be edge-coloured with colours $c_2, \ldots, c_{2n-1}$. An easy counting argument shows that each colour is missing from exactly one vertex. Therefore a total-colouring of G with the $d(G)+1$ colours $c_1, \ldots, c_{2n-1}$ can be obtained by colouring v_1, v_2, v_3 and the edges of F_1 with c_1, colouring the edges of $G \backslash F$ the same, and colouring each vertex v of $V(G) \backslash \{v_1, v_2, v_3\}$ with the colour missing at v.

Now suppose that $\bar{G}$ contains no K_3. Then by the argument used in the proof of Theorem 2, G is not conformable and so G is not of type 1. To see that G is of type 2, we argue as follows. Let $v_1 v_2$ and $v_2 v_3$ be edges of $\bar{G}$. The argument above can be used to show that $d(G)+1$ colours can be assigned to $V(G) \cup E(G)$ so that the total-colouring rules are obeyed everywhere except that v_1 and v_3 (which are adjacent) receive the same colour. Thus a total-colouring with $d(G)+2$ colours can be obtained by recolouring v_3 with a further colour.

THEOREM 4. *Let G be a graph of odd order with*

$$\delta(G) + \Delta(G) \geqslant \tfrac{3}{2}|V(G)| - 1.$$

Then

$$\chi_T(G) \leqslant \Delta(G) + 3.$$

COROLLARY. *Let G be a graph of odd order with*

$$\delta(G) \geqslant \tfrac{3}{4}|V(G)| - \tfrac{1}{2}.$$

Then

$$\chi_T(G) \leqslant \Delta(G) + 3.$$

The proof of Theorem 4 is very similar to the proof of Theorem 1, the difference being that we no longer have to restrict $\delta(G)$ sufficiently for Lemma 1 to be applied.

Proof of Theorem 4. We only indicate here the points where the proof of Theorem 4 differs from that of Theorem 1.

For $1 \leqslant j \leqslant 2n-\Delta$, the graphs G_j^+ are defined as before. If $j < 2n-\Delta$, then each vertex of G_j^+ has degree at least $\delta-(j-1) \geqslant \delta+\Delta-2n+2$. If $j = 2n-\Delta$ then in G_j^+, $v_1, \ldots, v_{j-1}$ have each been included in all but one of $F_1, \ldots, F_{j-1}$ and so have degree at least $\delta-(j-2) \geqslant \delta+\Delta-2n+2$. All other vertices of G_j^+ have at least this degree except possibly $v_{2n-\Delta}$. This may have degree as low as $\delta+\Delta-2n+1$. Provided $n \geqslant 3$, it follows from Lemma 3 that G_j^+ has a Hamiltonian cycle containing the path

$$v_j,\ v_{2(2n-\Delta)+j},\ v_{(2n-\Delta)+j},\ v^*.$$

(If $n \leqslant 2$, then $|V(G)| \leqslant 5$, and the total chromatic number conjecture is known to be true in this case; see, for example, **[16]**.)

The graph G^{**} has maximum degree $\Delta+1-(2n-\Delta-1) = 2\Delta-2n+3$ and can be edge-coloured with $2\Delta-2n+3$ colours. From this we go on to construct a total colouring of G from the edge-colouring of G^{**} as before. The only difference is that one more colour is used this time in the edge-colouring of G^{**}.

4. *Graphs of even order*

In this section we show that, for graphs of even order with $\delta(G) \geqslant \frac{3}{4}|V(G)|$, the total chromatic number conjecture holds. This is a stronger result than that obtained in the odd-order case, so it is surprising that the question of the classification of regular graphs of even order and high degree into type 1 and 2 graphs remains unresolved. In the case of the cocktail party graph, this question is resolved.

THEOREM 5. *Let G be a graph of even order with*

$$\delta(G)+\Delta(G) \geqslant \tfrac{3}{2}(|V(G)|-1).$$

Then

$$\chi_T(G) \leqslant \Delta(G)+2.$$

COROLLARY. *Let G be a graph of even order with*

$$\delta(G) \geqslant \tfrac{3}{4}(|V(G)|-1).$$

Then

$$\chi_T(G) \leqslant \Delta(G)+2.$$

Proof of Theorem 5. Let $|V(G)| = 2n$. The theorem is true if $n = 1$, so suppose that $n \geqslant 2$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Let the vertices of G be $v_1, \ldots, v_{2n}$ and, for $1 \leqslant j \leqslant 2n-\Delta-1$, let v_j and $v_{2n-\Delta-1+j}$ be non-adjacent. By Lemma 4, it follows that $2n-1-\Delta$ disjoint pairs of non-adjacent vertices exist.

From G form a graph G^* by introducing a vertex v^* and joining it to the $2\Delta-2n+2$ vertices $v_{2(2n-\Delta-1)+1}, \ldots, v_{2n}$. Then G^* has $2\Delta-2n+2$ vertices of degree at most $\Delta+1$, the vertex v^* has degree $2\Delta-2n+2$, and the remaining vertices have degree at most Δ. Since $2n-1 \geqslant \Delta$, it follows that $\Delta+1 \geqslant 2\Delta-2n+2$, and so the maximum degree of G^* is at most $\Delta+1$.

Now let $F_1, \ldots, F_{2n-\Delta-1}$ be edge-disjoint matchings of G^* such that, for

$$1 \leqslant j \leqslant 2n-\Delta-1,$$

F_j misses the three vertices $v_j, v_{2n-\Delta-1+j}, v^*$, but misses no further vertices.

We now proceed to show that these matchings do exist. We pick them out one by one. For $1 \leqslant j \leqslant 2n-\Delta-1$, suppose that $F_1, \ldots, F_{j-1}$ have been constructed. Let $G_j^* = G^* \backslash (F_1 \cup \ldots \cup F_{j-1})$, and let $G_j^+ = G_j^* \cup \{v_j v_{2n-\Delta-1+j}, v_{2n-\Delta-1+j} v^*\}$. Then, for $1 \leqslant j \leqslant 2n-\Delta-2$,

$$\delta-j+1 \geqslant \delta-(2n-\Delta-1)+1 = \delta+\Delta-2n+2 \geqslant \tfrac{1}{2}((2n+1)+2),$$

and, if $j = 2n-\Delta-1$, then the least degree in $G^* \backslash (F_1 \cup \ldots \cup F_{j-1})$, apart from that of v^*, is at least $\delta-(j-2) = \delta+\Delta-2n+3 \geqslant \frac{1}{2}((2n+1)+2)$. So in each case the least degree in G_j^+, apart from possibly that of v^*, is at least $\frac{1}{2}((2n+1)+2)$. If $d_{G_j^+}(v^*) \leqslant 3$ then it would follow that $2\Delta-2n+2 \leqslant 3$, so $\Delta \leqslant n$, which is impossible. Therefore $d_{G_j^+}(v^*) \geqslant 4$. Therefore, by Lemma 3, the graph G_j^+ contains a Hamiltonian cycle containing the path $v_j, v_{2n-\Delta-1+j}, v^*$. Therefore G_j^* contains a matching F_j which misses the vertices $v_j, v_{2n-\Delta-1+j}$ and v^*, but misses no other vertex. Iterating this gives the required matchings $F_1, \ldots, F_{2n-\Delta-1}$.

Let $G^{**} = G^* \backslash (F_1 \cup \ldots \cup F_{2n-\Delta-1})$. Since the degree of v^* is $2\Delta-2n+2$ and $\Delta+1-(2n-\Delta-1) = 2\Delta-2n+2$, the maximum degree of G^{**} is $2\Delta-2n+2$. Therefore, by Vizing's theorem, G^{**} can be edge-coloured with $2\Delta-2n+3$ colours.

From an edge-colouring of G^{**} with the $2\Delta-2n+3$ colours $c_{2n-\Delta}, \ldots, c_{\Delta+2}$, we form a total-colouring of G with the $\Delta+2$ colours $c_1, \ldots, c_{\Delta+2}$ as follows. Each edge of G which is also an edge of G^{**} receives the same colour. For

$$2(2n-\Delta-1)+1 \leqslant j \leqslant 2n,$$

the vertex v_j receives the colour of the edge $v_j v^*$. For $1 \leqslant j \leqslant 2n-\Delta-1$, the two vertices v_j and $v_{(2n-\Delta-1)+j}$ each receive the colour c_j, and the edges of $F_j \cap E(G)$ also receive the colour c_j. It is easy to check that this is a total-colouring of G.

This proves Theorem 5.

THEOREM 6. *Let G be a cocktail-party graph (that is, K_{2n} less a 1-factor). Then G is of type 1 if and only if $|V(G)| \neq 4$.*

Proof. If $|V(G)| = 4$. then $G = C_4$ which is of type 2. If $|V(G)| \geqslant 6$ then Theorem 6 follows from a result in **[1]** by Andersen and Hilton. (It also follows from an unpublished result of Häggkvist, and probably there are other proofs as well.)

5. *Graphs of high minimum degree*

Finally, we summarise the results which we now know are true for all graphs of high minimum degree.

THEOREM 7. *If G is a graph with $\delta(G) \geqslant \frac{5}{6}(|V(G)|+1)$ then the total chromatic number conjecture is true for G.*

Proof. This follows from the corollaries to Theorems 1 and 5.

THEOREM 8. *If G is a graph with $\delta(G) \geqslant \frac{3}{4}(|V(G)|-1)$ then $\chi_T(G) \leqslant d(G)+3$.*

Proof. This follows from the corollaries to Theorems 4 and 5.

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Remarks added in proof. A number of points have recently come to our attention which ought to be mentioned here. Recently the second author and Hind **[23]** have proved that if $\Delta(G) \geqslant \frac{3}{4}|V(G)|$ then $\chi_T(G) \geqslant \Delta(G)+2$. The bound $\chi_T(G) \geqslant \Delta(G)+t+1$ (which is marginally weaker than the one of the first author and Häggkvist **[5]** was proved independently by McDiarmid and Reed (see **[24]**). The bounds of Lemmas 1 and 2 were recently proved independently by Niessen and Volkman **[25]**. An isolated counterexample to Conjecture 1 has been discovered by Bor-Liang Chen and Hung-Lin Fu **[22]**.

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